Quantum phase transition in an array of coupled superconducting quantum dots with charge frustration

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The quantum phase transition in two capacitively coupled arrays of superconducting quantum dots (SQD) by considering the presence of gate voltage in each superconducting island is presented. We show explicitly that the cotunneling process associated with two coupled SQD arrays, near the maximum-charge frustration line is not sufficient to explain the correct quantum phases with physically consistent phase boundaries. We consider an extra cotunneling process along each chain to explain the correct quantum phases with physically consistent phase boundaries. There is no evidence of supersolid phase in our study. The Bethe ansatz and Abelian bosonization method is used to solve the problem.

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Josephson-junction arrays have attracted considerable interest in the recent years owing to their interesting physical properties.[1–14] Currently such arrays can be fabricated in restricted geometries both in one and two dimensions with well-controlled parameters.[15–18] At the same time this is one of the paradigms to study the physics of quantum phase transition.[1,7,8] The Coulomb charging energy of the system is due to the small capacitance of the grains or junction that dominate strong quantum phase fluctuations of the superconducting (SC) order parameter and may drive the system into the insulating state at zero or very low temperature. The quantum fluctuations are controlled by the parameters of the system such as charging energy of the superconducting quantum dots (SQD) and Josephson coupling (E_J) between them.

The universality class of these transitions in two dimensions has already been investigated in detail.[10–14] Nevertheless there are also quantum phase transition scenario that can take place in a Josephson-junction ladder system as shown in the Fig. 1, where we consider two capacitively coupled one-dimensional SQD arrays. In this simplest two-leg ladder system, one can predict different quantum phases due to the interplay between Coulomb charging energy and E_J, where the role of cotunneling effect is also explicitly studied. For large Coulomb charging energy, sequential tunneling is not energetically favored process in the system. The major tunneling process in the system occurs via the cotunneling process[19] and its origin is quantum mechanical. In this process, tunneling occurs through the virtual state with energy equal to the on-site Coulomb charging energy. An appropriate cotunneling process gives the correct quantum phases in the system. Otherwise, it leads to the wrong analysis of the system as we have found in Ref. 20. In our previous studies emphasis was on the importance of next-nearest-neighbor E_J (Refs. 7 and 8) for the one-dimensional SQD array.

In this Brief Report we study the quantum phase transition of capacitively coupled two arrays of SQD. We consider the presence of gate voltage in each SQD which introduces the charge frustration in the system. At the charge frustration line the Coulomb charging energy of the system is degenerate for the difference of one Cooper pair in the island. It is also referred in the literature as a charge degeneracy point. Our main motivation is to study the different quantum phases around the maximum-charge frustration line. Choi *et al.*[20] have shown that in a coupled chain the major transport along both the chains occurs via cotunneling of the electron-hole pairs. In this Brief Report, we address three major issues which are absent in the previous study.[20] (1) There are two kinds of Luttinger liquid (LL) phase (described as RL1 and RL2) but they have obtained only one. (2) Incorrect quantum phase analysis and physically inconsistent phase boundaries due to lack of correct analytical derivations and physical interpretations. (3). There is no evidence of supersolid (SS) phase in this model.

We show explicitly that a single cotunneling process in the two-leg SQD ladder system is not sufficient to produce the correct quantum phases with physically consistent phase boundaries. We also show that the cotunneling process along each chain is also necessary for the correct phase boundaries. In our model system, superconducting islands are connected with E_J. The charging energies along each array are the on-site Coulomb charging energy due to the self-capacitance, $E_0 = \frac{e^2}{2C_0}$ and the junction charging energy, $E_1 = \frac{e^2}{2C_1}$ due to junction capacitance C_1. The arrays are coupled to each other via the capacitance C_I which contributes charging energy, $E_I = \frac{e^2}{2C_I}$. But there is no Josephson coupling between the two arrays. The array capacitances are small, which lead to $E_J \ll E_0, E_1$. Therefore each array is in the insulating phase in

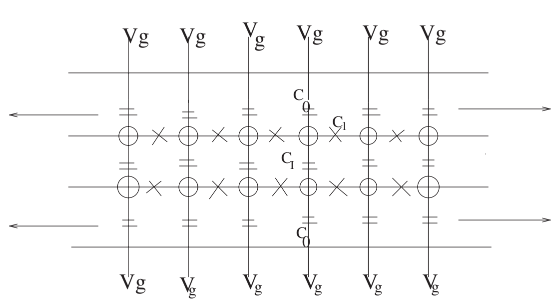

FIG. 1. This is the schematic of our model system in which two arrays of superconducting quantum dots are connected through the capacitance, C_I. C_0 is the self-capacitance of the dot and C_1 is the junction capacitance. In our superconducting circuit, small circles denote the dots and crosses denote the Josephson junctions. V_g is the gate voltage applied in each dot.

the absence of cotunneling effect. Here we consider the limit $E_I \ll E_0, E_1$. The applied gate voltage induces a charge in every superconducting island by an amount, $n_g = \frac{C_0 V_g}{2e}$, this applied gate voltage introduces the charge frustration in the system. We do the quantum phase analysis near the maximum-charge frustration line ($|n_g - N - 1/2| \ll 1$), and we obtain several interesting quantum phases around this line which have not been noticed in the previous studies.[5,6,20] The Hamiltonian of the system is given by

$$H = 2e^2 \sum_{ll',xx'} [n_l(x) - n_g] C^{-1}_{l,l'}(x,x')[n_l(x') - n_g] - E_J \sum_{l,x} \cos[\theta_l(x) - \theta_l(x+1)], \quad (1)$$

where $n_l(x)$ is the number of Cooper pairs and $\theta_l(x)$ is the phase of the superconducting order parameter at the site x of the lth array. $n_l(x)$ and $\theta_l(x)$ are the quantum mechanically conjugate variables. One can write the capacitance matrix in the block form

$$C^{-1}_{l,l'}(x,x') = C(x,x')A + \delta_{x,x'} C_I B, \quad (2)$$

where

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$B = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix},$$

$$C(x,x') = C_0 \delta_{x,x'} + C_1[2\delta_{x,x'} - \delta_{x,x'+1} - \delta_{x,x'-1}], \quad (3)$$

$$H = H_c^{(0)} + H_c^{(1)} + H_J \quad (4)$$

with the components

$$H_c^{(0)} = U_0 \sum_x [n_+(x) - 2n_g]^2 + V_0 \sum_x [n_-(x)]^2, \quad (5)$$

$$H_c^{(1)} = U_1 \sum_x [n_+(x) - 2n_g][n_+(x+1) - 2n_g] + V_1 \sum_x n_-(x) n_-(x+1), \quad (6)$$

$$H_J = -E_J \sum_{l,x} \cos[\theta_l(x) - \theta_l(x+1)], \quad (7)$$

where $n_\pm(x) = n_1(x) \pm n_2(x)$ and the coupling strengths are given by $U \sim 2E_0$, $U_1 = 4\frac{C_I}{C_0}$, $V_0 \sim E_I$, and $V_1 \sim \frac{C_1 C_I}{E_I}$. The first term of the Hamiltonian $H_c^{(0)}$ contains an important message regarding the difference between the charging energy near the maximum-charge frustration line and the particle-hole symmetric line ($n_g = 0$). The charge configurations which do not satisfy the condition for $n_+(x) = 1$ at the maximum-charge frustration line generate a gap in the excitation spectrum on the order of on-site charging energy. The ground state of $H_c^{(0)}$ separated from the excited state by the gap on the order of E_I. This excited state has twofold degeneracy for every values of x, corresponding to $n_-(x) = \pm 1$. It is convenient to work in the charge configuration with $n_+(x) = 0$ and $n_-(x) = \pm 1$ is termed as a reduced Hilbert space of the problem. Presence of finite E_j lifts this degeneracy and the ground state of $H_c^{(0)}$ is mixed with the state $n_-(x) = \pm 2$. Now the relevant reduce Hilbert space, $n_+(x) = 0$ and $n_-(x) = 0, \pm 2$. Here we mention the effective model of our system, wherein we follow Ref. 20. They have found the effective Hamiltonian up to the second order in $\frac{E_J}{E_0}$

$$H_{eff} = P\left[H + H_J \frac{1-P}{E - H_C^0} H_J\right] P. \quad (8)$$

Here P is the projection operator onto the reduced space. The final effective Hamiltonian is

$$H_{eff} = \gamma J \sum_x S^z(x) S^z(x+1), \frac{-J}{2} \sum_x [S^+(x) S^-(x+1) + \text{H.c.}], \quad (9)$$

where the XY component of exchange interaction and the Z component of exchange anisotropy are, respectively, $J = \frac{E_J^2}{4E_0}$ and $\gamma = \frac{16\lambda^2 E_I^2}{E_J^2}$. No analytical expression for λ^2 is given in Ref. 20. Here we estimate $\lambda^2 = \frac{C_1 E_0}{C_I E_I}$. Pseudospin operators of the above Hamiltonian are the following:

$$S^z(x) = P \frac{n_1(x) - n_2(x)}{2} P,$$

$$S^+(x) = P e^{-i\theta_1(x)} (1-P) e^{-i\theta_2(x)} P,$$

$$S^-(x) = P e^{-i\theta_2(x)} (1-P) e^{-i\theta_1(x)} P. \quad (10)$$

Now we explain the pseudospin of the Hamiltonian in terms of charge representation from where we have started. The first term of the Hamiltonian originates from the first term of Eq. (9), which is the Coulomb charging energy of the SQD, in the form of $n(x)$. The second term of the Hamiltonian originates from the cotunneling process, i.e., from the second term of Eq. (9). Choi *et al.*[20] have obtained the Hamiltonian in an effective one-dimensional spin-chain Hamiltonian.

Our analytical analysis of this model Hamiltonian is based on Luttinger liquid formalism and Bethe ansatz: at first we express effective spin-chain Hamiltonian in term of spinless-fermion Hamiltonian by using the Jordan-Wigner transformation. In Jordan-Wigner transformation the relation between the spin and the electron creation and annihilation operators are

$$S^z(x) = \psi^\dagger(x)\psi(x) - 1/2,$$

$$S^-(x) = \psi(x) \exp\left[i\pi \sum_{j=-\infty}^{x-1} n_j \right], \quad (11)$$

where $S^+ = (S^-)^+, n(x) = \psi^\dagger(x)\psi(x)$ is the fermion number at the site x

$$H_{eff} = -\gamma E_J \sum_x [\psi^\dagger(x)\psi(x) - 1/2],$$

$$[\psi^\dagger(x+1)\psi(x+1) - 1/2],$$

$$\frac{-J}{2}\sum_x [\psi^\dagger(x+1)\psi(x) + \text{H.c.}]. \quad (12)$$

In order to study the continuum-field theory of these Hamiltonians, we recast the spinless-fermion operators in terms of field operators by the relation $\psi(x)=[e^{ik_F x}\psi_R(x) + e^{-ik_F x}\psi_L(x)]$, where $\psi_R(x)$ and $\psi_L(x)$ describe the second quantized fields of right- and left-moving fermions, respectively.[21] We want to express the fermionic fields in terms of bosonic fields by the relation $\psi_r(x) = \frac{U_r}{\sqrt{2\pi\alpha}} e^{-i[r\phi(x)-\theta(x)]}$, where r represents the chirality of the fermionic fields, right (1) or left movers (−1). The operators U_r commute with the bosonic field. U_r of different species commute and U_r of the same species anticommute. ϕ field corresponds to the quantum (bosonic) fluctuations of spin and θ is the dual field of ϕ. They are related by the relation $\phi_R = \theta - \phi$ and $\phi_L = \theta + \phi$. After doing the continuum-field theory exercise, H_{eff} becomes

$$H_{eff} = H_0 + \frac{\gamma J}{2\pi\alpha^2}\int dx{:}\cos[4\sqrt{K_1}\phi(x)], \quad (13)$$

where $H_0 = \frac{v}{2\pi}\int dx[(\partial_x\theta)^2 + (\partial_x\phi)^2]$ is the noninteracting part of the Hamiltonian. The collective velocity of the system (v) and K_1 are the two LL parameters. We use exact Bethe ansatz solution to calculate

$$K_1 = \frac{\pi}{\pi + 2\sin^{-1}\gamma}. \quad (14)$$

For $J < \gamma J$ with relatively small applied gate voltage, the antiferromagnetic Ising interaction dominates the physics of anisotropic Heisenberg chain. When the applied gate voltage is large the chain is in the ferromagnetic state. In the language of interacting Cooper pairs, the Neel phase is the charge-density wave (CDW) phase with period 2, i.e., there is only one Cooper pair in every two sites. For the ferromagnetic phase, the system is in the Mott insulating (MI) phase. The emergence of two LL phases can be ascribed to the following reasons: RL1 and RL2, occur due to the commensurate-incommensurate transition and the criticality of the Heisenberg XY model, respectively. The system is either in RL1 for $K < 1/2$ or in RL2 for $K > 1/2$. The physical significance of RL1 is that the coupling term is relevant but the applied gate voltage on SC island breaks the CDW gapped phase, whereas in RL2, system is gapless for $K_1 > 1/2$.

In Fig. 2, we present the quantum phases of our study as a function of K. We apply the Luther-Emery trick[22] in the massive phase of sine-Gordon field theory to evaluate K_1($=1/4$) at the phase boundary between the CDW and RL1 and also for CDW and RL2. RL1 phase is not explicit in the figure because we achieve this phase only under the application of gate voltage when it exceeds the gap of CDW state. The value of K_1 at the phase boundary between RL2 and CDW phase is 1/4. From the analysis of K_1 we obtain $16\lambda^2 E_I^2 = -E_J^2$, this condition is unphysical because λ, E_I, and E_J are positive quantities. The value of K is 1 at the phase boundary between the RL2 and the superconducting phase. The analysis yields the condition $16\lambda^2 E_I^2 = 0$, which is again unphysical. Therefore, there is no evidence of superconductivity in this model Hamiltonian.

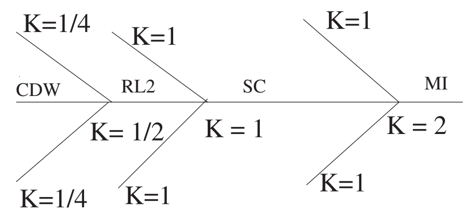

FIG. 2. The one-dimensional schematic for quantum phases as a function of Luttinger liquid parameters (K) and also values of K at the phase boundaries. CDW is the charge-density-wave phase. RL2 is the repulsive Luttinger liquid phase of second kind. SC is the superconducting phase. The limit $K=2$ can be achieved only when the system is in the first-order commensurability, i.e., each dot is occupied with integer number of Cooper pairs. MI is the Mott insulating phase of the system. The presence of gate voltage is not explicit in K, therefore the RL1 phase does not appear in the figure, we only achieve this phase only when the applied gate voltage exceeds the CDW gap.

To get the correct physical behavior between the different quantum phases, one has to consider the cotunneling processes along each chain separately. Our calculation shows that to get an attractive interaction between the Jordan-Wigner (spinless) fermions, one has to consider higher order expansion in $\frac{E_{J1}}{E_{C0}}$. This higher order expansion leads to the virtual state with energies exceeding E_{C0}. In this second-order process, extra contribution appears as

$$H_{extra} = \frac{-3E_J^2}{4E_0}\sum_x S^z(x)S^z(x+1)$$

$$-\frac{E_J^2}{E_0}\sum_x [S^\dagger(x+2)S^-(x) + \text{H.c.}] \quad (15)$$

(Refs. 6–8 and 23). The total effective Hamiltonian of the system under the combined cotunneling process is

$$H_{eff} = \left(\gamma J - \frac{3E_J^2}{4E_0}\right)\sum_x S^z(x)S^z(x+1)$$

$$-\frac{J}{2}\sum_x [S^\dagger(x)S^-(x+1) + \text{H.c.}]$$

$$-\frac{E_J^2}{E_0}\sum_x [S^\dagger(x+2)S^-(x) + \text{H.c.}]. \quad (16)$$

After doing the quantum-field-theory analysis, we get the effective Hamiltonian

$$H_{eff}=H_0+\frac{\gamma J-\frac{3E_J^2}{4E_0}}{2\pi\alpha^2}\int dx{:}\cos[4\sqrt{K_2}\phi(x)], \qquad (17)$$

LL parameter of H_{eff} is

$$K_2=\sqrt{\left[\frac{J}{J+4/\pi\left(\gamma J-\frac{3E_J^2}{4E_0}\right)}\right]}. \qquad (18)$$

We notice from Fig. 2 that the value of K_2 is 1/4 at the phase boundary between RL2 and CDW phase. The parametric condition at this phase boundary is $\frac{16\lambda^2E_I^2}{E_J^2}=\frac{15\pi}{4}+3$. This condition is consistent physically. Therefore we prove the importance of this extra cotunneling process. We see from Fig. 2 that value of the K_2 is 1 at the phase boundary between the RL2 and superconducting phase. The parametric analysis for this value of K_2 yields $16\lambda^2E_I^2=3E_J^2$, and hence we obtain, $E_J^2=8\lambda^2E_I^2$. This is a physically realizable condition for the phase boundary. It is clear from Fig. 2 that there is no simultaneous presence of CDW phase and SC phase. SC phase occurs when $K>1$ and CDW phase occurs when $1/2<K<1$. Therefore there is no evidence of SS phase for this model system.

We have found the all correct quantum phases of this model Hamiltonian based on rigorous analytical calculations. We have obtained the condition for physically consistent phase boundaries which were absent in the previous studies. There is no evidence of supersolid phase.

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